

Comments on the symmetries of various supergravities in 5 dimensions

*C. Rugina*¹
Physics Department
University of Bucharest
Bucharest, Romania
and
Department of Theoretical Physics
IFIN-HH
Str. Reactorului no. 30
P.O. Box MG-6
Postcode 077125
Bucharest - Magurele, Romania
and
Department of Physics
Imperial College
London, SW7 2AZ
UK

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ABSTRACT

We investigate the hidden symmetries of various D=5 minimal gauged supergravities which admit a Killing-Maxwell system in the sense of Carter, such as the Chong-Cvetič-Lü-Pope black hole spacetime and a 5-dimensional minimal gauged supergravity endowed with a Sasaki structure deformed by torsion. We note that when an electromagnetic tensor is present and an associated Killing-Maxwell system can be constructed in the sense of Carter, the Killing-Maxwell field becomes a PCKY (primary conformal Killing-Yano) tensor or a PGCKY (primary generalized conformal Killing-Yano) tensor, the latter in the presence of torsion. We study some of the properties of the Chong-Cvetič-Lü-Pope black hole, we construct two more Stäckel-Killing tensors and we prove that the associated three generalized Killing-Yano (GKY) tensors form a hyper-Kähler deformed by torsion structure for this spacetime. Moreover the three associated modified Dirac-type operators (associated with the three GKY) generate a superalgebra of modified operators. We also explicitly construct a Killing spinor for the above mentioned specific 5 dimensional minimal gauged supergravity solution that is endowed with a Sasaki structure deformed by torsion.

¹cristina.rugina11@imperial.ac.uk

1 Introduction

Symmetries in nature have long been useful in determining constants of motion and hence in helping solve equations of motion. Hidden symmetries, the symmetries of the phase space, together with spacetime symmetries bring insight into the evolution of a system in curved spacetime and are characterized by symmetrical Stäckel-Killing and antisymmetrical Killing-Yano tensors[1], while the latter symmetries are driven by Killing vectors.

There is a long history of separation of variables of Hamilton-Jacobi, Klein-Gordon, and Dirac equations[2,3,4,5,6,7,8,9,10,11,12] against various background spacetimes, starting with the Kerr spacetime in four dimensions and continuing with higher dimensional black hole spacetimes, the work cited above having a particular emphasis on the most general Kerr-AdS-NUT spacetime. All these with the miraculous help of Stäckel-Killing and Killing-Yano tensors, which build up constants of motion, which in turn lead to complete integrability of the equations. Another important object in describing hidden and spacetime symmetries is the PCKY and more generally the CKYs (conformal Killing-Yano tensors) [13,14,15,16]. The PCKY generates in higher dimensional spacetimes towers of Killing-Yano and Stäckel-Killing tensors.

The group structure and algebras generated by Dirac-type operators in spacetimes that are torsionless have been thoroughly studied [17,18,19] and these operators constructed with the use of Killing-Yano tensors turn out not to have any anomalies at the quantum level. These operators generate standard and non-standard supersymmetries of the spinning point particle evolving in the respective spacetimes.

One other important aspect is the fact that there exist geometrical dualities that map spacetimes with torsion in duals that are torsionless[20, 21, 22]. If a Killing-Yano structure exists in the spacetime with torsion, then it becomes the vielbein of the torsionless dual spacetime, while the vielbein of the spacetime with torsion becomes the Killing-Yano structure of the torsionless dual.

A lot of progress[23] has been accumulated since the seminal paper of Myers and Perry[24], where the metrics describing the isolated, vacuum, rotating higher-dimensional Kerr black holes were derived. Going forward in time we reach the derivation of the most general black-hole in D=5 minimal gauged supergravity spacetime metric [25] which is one of the objects of attention in this current paper. We cannot do justice here to the entire scientific effort of determining the metrics of various black holes in higher dimensions since 1986 on.

Solving the equations of motion in spacetimes of black holes with gauge fields is aided by the so-called generalized Killing-Yano, generalized Stäckel-Killing and generalized conformal Killing-Yano tensors corresponding to symmetries with torsion. There is an ample span of study of these tensors in the literature ranging from the D=5 minimal gauged supergravity spacetime [26,27], to the study of the black hole spacetimes in the framework of string theory [28,29,30].

In this paper we present some relevant calculus with torsion in section 2; then in section 3 we introduce some results obtained by Carter in the context of the Kerr-Newman black-hole, dubbed the Killing-Maxwell system and we also review some results obtained previously in [17] where Dirac-type operators were constructed to describe hidden symmetries (here we write them out specifically in the framework of the Killing-Maxwell system of the Kerr-Newman black hole). We then study in section 4 the Killing-Maxwell system in the context of D=5 minimal gauged supergravity and we review the structure of the Chong-Cvetič-Lü-Pope (CCLP) black hole spacetime and the fact that this spacetime has the structure of a hyper-Kähler spacetime deformed by torsion. We find two more generalized Stäckel-Killing tensors for this spacetime. Finally, in section 5 we explicitly extract a Killing spinor for a specific manifold endowed with a Sasaki structure deformed by torsion, that is a solution of minimal gauged supergravity in 5 dimensions.

2 Some calculus relations in the presence of torsion

Let's introduce by definition a few concepts: the Killing-Yano tensor, the Stäckel-Killing tensor and the conformal Killing-Yano tensor on a pseudo-Riemannian manifold (\mathcal{M}, g) .

Definition. A differential p-form $Y \in \Omega^p(\mathcal{M})$ is called a Killing-Yano tensor if $\nabla Y \in \Omega^{p+1}(\mathcal{M})$ that is, if $\nabla_\mu Y_{\alpha_1 \dots \alpha_p}$ is totally antisymmetric.

Definition. A totally symmetric contravariant tensor K of rank p over \mathcal{M} is called a Killing tensor if $\nabla^{(\mu} K^{\alpha_1 \dots \alpha_p)} = 0$.

Definition. A p-form $Y \in \Omega^p(\mathcal{M})$ is called a conformal Killing-Yano tensor if:

$$\nabla_X Y = \frac{1}{p+1} X \lrcorner dY - \frac{1}{n-p+1} X^* \wedge d^* Y \quad (1)$$

where X is a vector field and X^* is the metric dual of vector X.

These definitions also hold in the case of a manifold with torsion and the tensors are called generalized, one just has to write the meaningful connection in this case. Let's now take a look at some calculus with torsion (see [31]-[35]). Let T be a 3-form on a pseudo-Riemannian manifold (\mathcal{M}, g) and e_a an orthonormal frame such that $g(e_a, e_b) = \delta_{ab}$. Then if X, Y are vector fields then we define the Levi-Civita connection as:

$$\nabla_X^T Y = \nabla_X Y + \frac{1}{2} T(X, Y, e_a) e_a, \quad (2)$$

T is assimilated with torsion and we shall also use the T=2A notation for torsion. For a p-form ω the covariant derivative is:

$$\nabla_X^A \omega = \nabla_X \omega - (X \lrcorner e_b \lrcorner A) \wedge (e_b \lrcorner \omega) \quad (3)$$

with the explicit formula for a 2-form being:

$$\nabla_\mu^A Y_{\nu\rho} = \nabla_\mu Y_{\nu\rho} - 2A_{\sigma\mu[\nu} Y^{\sigma]}_{\rho]}. \quad (4)$$

Note that the spinor covariant derivative with torsion can be written out as:

$$D_\mu^A = D_\mu + \frac{1}{12} \gamma^\nu \gamma^\rho A_{\mu\nu\rho}. \quad (5)$$

Hence the Dirac operator with torsion is:

$$D_\mu^A \gamma^\mu = D_\mu \gamma^\mu + \frac{1}{12} \gamma^\mu \gamma^\nu \gamma^\rho A_{\mu\nu\rho}. \quad (6)$$

The Ricci relation with torsion for a Killing-Yano 2-form is as follows:

$$\nabla_\alpha^A \nabla_\beta^A Y_{\mu\nu} = -\frac{3}{2} R^\lambda_{\alpha\beta[\mu} Y_{\nu]\lambda} - 2A^{|\lambda|}_{\beta[\alpha} \nabla_{|\lambda|}^A Y_{\mu]\nu}. \quad (7)$$

The square of the Dirac operator as a function of the torsion T is:

$$D^{2T} = -\Delta^T - \frac{dT}{4} - \frac{s}{4} - \frac{\|T\|^2}{24}, \quad (8)$$

where

$$\Delta^T = \nabla_{X_a}^T \nabla_{X^a}^T + \nabla_{\nabla_{X_a}^T X^a}^T, \quad (9)$$

and s is the scalar curvature of the connection with torsion-

$$s = -X^a \lrcorner R(X_a, X_b) e^b. \quad (10)$$

The curvature operator is defined as usual:

$$R(X, Y)\omega = (\nabla_X^T \nabla_Y^T - \nabla_Y^T \nabla_X^T - \nabla_{[X, Y]}^T)\omega. \quad (11)$$

So the spinor covariant derivatives commutator is:

$$[D_\mu^A, D_\nu^A]\Psi = \frac{1}{8} R_{\alpha\beta\mu\nu} [\gamma^\alpha, \gamma^\beta] \Psi - A^\lambda_{\mu\nu} D_\lambda^A \Psi. \quad (12)$$

3 Hidden symmetries of the Kerr-Newman space-time

The minimal gauged supergravity in 5 dimensions spacetime together with its symmetries has been recently studied [26] and in this framework- also the most general known Chong-Cvetič-Lü-Pope black hole solution[25]. Previous work is dating back to 1987 and is done by Carter[36,37], who investigated the solutions and symmetries of its lower-dimensional cousin, the Kerr-Newman black hole.

Carter reached the conclusion that there exists a Killing-Maxwell electromagnetic system defined by the following equation for the 4-dimensional electromagnetic potential (here we used Carter's notations, in that semi-colon means taking the covariant derivative):

$$\hat{A}_{[\mu;\nu];\rho} = 2\frac{4\pi}{3}\hat{j}_{[\mu}g_{\nu]\rho}, \quad (13)$$

where g is the spacetime metric and \hat{j} the current, which is a Killing vector. The Killing-Maxwell electromagnetic system obeys regular Maxwell equations, since the Maxwell field is defined as usual and respects Maxwell's laws-

$$\hat{F}_{;\rho}^{\rho\mu} = 4\pi\hat{j}^\mu \quad (14)$$

and

$$\hat{F}_{[\mu\nu;\rho]} = 0. \quad (15)$$

Further, Carter mentions that the Hodge dual of the Killing-Maxwell electromagnetic field is a Killing-Yano tensor of rank 2. Although he doesn't state it directly, it follows that \hat{F} is a PCKY tensor - as defined for instance by Kubizňák in his thesis, rel. (3.7) in [13]- since it satisfies equations (13) and (14). The fact that \hat{F} is closed follows from Maxwell's laws.

The Hodge dual of the Killing-Maxwell electromagnetic field consequently determines a Dirac-type operator, that anti-commutes with the Dirac operator in the Kerr-Newman spacetime, according to [17]:

$$Q_{*\hat{F}} = \gamma^\mu * \hat{F}_\mu^\nu D_\nu - \frac{1}{6} \gamma^\mu \gamma^\nu \gamma^\rho \nabla_\mu * \hat{F}_{\nu\rho}. \quad (16)$$

This operator corresponds to a quantum (non-anomalous) hidden symmetry for the spinning point particle in Kerr-Newman spacetime which is - as stated in [17]- an additional non-generic supersymmetry of the particle. It is a remarkable fact that $*\hat{F}$ generates a supersymmetry and it points out the subtle connection between the symmetries of the Killing-Maxwell electromagnetic field in curved spacetime and this supersymmetry and further- between spin and electric charge. In the 4-dimensional spacetime, the PCKY generates one Killing-Yano tensor, which is $*\hat{F}$ and one Stäckel-Killing tensor, K :

$$K_{\mu\nu} = (*\hat{F})_{\mu\rho} (*\hat{F})_\nu^\rho. \quad (17)$$

4 Hidden symmetries of the D=5 minimal gauged supergravity spacetime with a Killing-Maxwell system

In the cousin spacetime, D=5 minimal gauged supergravity, the Hodge dual of the electromagnetic field plays the role of torsion as evidenced in [26]. Here we

are going to focus on the case when $*F$ is part of the Killing-Maxwell system, a generalization to 5 dimensions of the work set forth by Carter. This means we are going to identify the PGCKY of the 5-dimensional spacetime, of which $*F$ is a Killing-Yano tensor, with F (the Killing-Maxwell electromagnetic field), so this is a spacetime that is a bit different than the one described in [26]. The definition of a Killing-Maxwell electromagnetic field in D=5 supergravity is:

$$\nabla_\rho F_{\mu\nu} = 2\pi g_{\rho[\mu} j_{\nu]} \quad (18)$$

and together with:

$$\nabla_\rho F^{\rho\mu} = 4\pi j^\mu \quad (19)$$

form a Killing-Maxwell system that obeys Einstein-Maxwell's laws. According to [26] the PGCKY of the D=5 minimal gauged supergravity spacetime has the definition:

$$\nabla_\rho h_{\mu\nu} = 2g_{\rho[\mu} \xi_{\nu]} - \frac{1}{\sqrt{3}} (*F)_{\rho\sigma} h^\sigma{}_{\nu]}. \quad (20)$$

If in our case h is indeed F , hence, equation (20) becomes:

$$\nabla^A h_{\mu\nu} = 2g_{\rho[\mu} \xi_{\nu]}. \quad (21)$$

Note that if in the equation above we notate h by F and we set $j = \pi\xi$ then indeed the definition in 5-dimensions of a Killing-Maxwell electromagnetic field and that of a PGCKY coincide and hence our assumption that for the five-dimensional supergravity endowed with a Killing-Maxwell system, such that $*F$ is Killing-Yano, then F and h coincide is true. According to the theory of Killing-Yano tensors, $*F$ the Killing-Yano tensor is the Hodge dual of the Killing-Maxwell electromagnetic field and which also plays the role of torsion in this spacetime. It also generates the Stäckel-Killing tensor:

$$K_{\mu\nu} = (*F)_{\mu\rho\sigma} (*F)^{\nu\rho\sigma}. \quad (22)$$

If we now take a look at the Chong-Cvetič-Lü-Pope black hole (CCLP) in D=5 minimal gauged supergravity framework, with the following notations:

$$g = \sum_{\mu=x,y} (\omega^\mu \omega^\mu + \tilde{\omega}^\mu \tilde{\omega}^\mu) + \omega^\epsilon \omega^\epsilon, \quad (23)$$

$$A = \sqrt{3}(A_q + A_p). \quad (24)$$

And-

$$\omega^x = \sqrt{\frac{x-y}{4X}} dx, \quad \tilde{\omega}^x = \frac{\sqrt{X}(dt + yd\phi)}{\sqrt{x(y-x)}}, \quad (25)$$

$$\omega^y = \sqrt{\frac{y-x}{4Y}} dy, \quad \tilde{\omega}^y = \frac{\sqrt{Y}(dt + xd\phi)}{\sqrt{y(x-y)}}, \quad (26)$$

$$\omega^\epsilon = \frac{1}{\sqrt{-xy}}[\mu dt + \mu(x+y)d\phi + xy d\psi - yA_q - xA_p], \quad (27)$$

$$A_q = \frac{q}{x-y}(dt + yd\phi), \quad A_p = \frac{-p}{x-y}(dt + xd\phi), \quad (28)$$

and

$$X = (\mu + q)^2 + Ax + CX^2 + \frac{1}{12}\Lambda x^3, \quad (29)$$

$$Y = (\mu + p)^2 + By + Cy^2 + \frac{1}{12}\Lambda y^3. \quad (30)$$

Please note that it was proven in [38] that CCLP is the unique minimal gauged supergravity spacetime with torsion such that the torsion tensor is both closed ($d^T T = 0$) and co-closed ($\delta^T T = 0$). We then find following [26] that, F, the PGCKY in our case is:

$$F = \sqrt{-x}\tilde{\omega}^x \wedge \omega^x + \sqrt{-y}\tilde{\omega}^y \wedge \omega^y \quad (31)$$

and the corresponding Killing tensor-

$$K = y(\omega^x \omega^x + \tilde{\omega}^x \tilde{\omega}^x) + x(\omega^y \omega^y + \tilde{\omega}^y \tilde{\omega}^y) + (x+y)\omega^\epsilon \omega^\epsilon. \quad (32)$$

Note that F, the Killing-Maxwell field above, is different from the electromagnetic field in the CCLP black hole spacetime, as cited in [26]. Note that K is directly involved in separating the Hamilton-Jacobi and Klein-Gordon equations in this spacetime.

We can now proceed to write down a couple of new GCCKY (generalized closed conformal Killing-Yano) tensors for the above metric:

$$h_{ab} = 4\omega_{[a}(\partial_\psi)_{b]} \quad (33)$$

which obeys the equations ($\xi = (\partial_\psi)$)

$$\nabla_c h_{ab} = 2g_{c[a}\xi_{b]} \quad (34)$$

and

$$\xi_b = \frac{1}{D-1}\nabla_d h^d{}_{b}. \quad (35)$$

In a similar way we can retrieve:

$$h_{ab} = 4\omega_{[a}(\partial_\phi)_{b]}. \quad (36)$$

Consequently the tensors above generate 2 new Stäckel-Killing tensors via the following relation:

$$K_{ab} = h_{ac}h_b^c - \frac{1}{2}g_{ab}h^2 \quad (37)$$

which are -

$$K_{ab}^\psi = 16\omega_{[a}(\partial_\psi)_c]\omega_{[b}(\partial_\psi)^{c]} - 4g_{ab}(\omega_d\omega^d(\partial_\psi)_c(\partial_\psi)^c - \omega_d\omega^c(\partial_\psi)_c(\partial_\psi)^d), \quad (38)$$

respectively

$$K_{ab}^\phi = 16\omega_{[a}(\partial_\phi)_c]\omega_{[b}(\partial_\phi)^{c]} - 4g_{ab}(\omega_d\omega^d(\partial_\phi)_c(\partial_\phi)^c - \omega_d\omega^c(\partial_\phi)_c(\partial_\phi)^d). \quad (39)$$

Also for spacetimes with $D \geq 6$ where an electromagnetic field is present and a Killing-Maxwell system can be constructed, the PGCKY is, naturally, in these spacetimes still the Killing-Maxwell electromagnetic tensor, F , which generates a tower of Killing-Yano and Stäckel-Killing tensors as follows:

$$F^{(j)} = F \wedge \dots \wedge F. \quad (40)$$

Above the wedge is taken j times, $F^{(j)}$ is a $(2j)$ -form and $F^{(1)} = F$. $F^{(j)}$ is a set of $(n-1)$ non-vanishing closed CKY (conformal Killing-Yano) tensors, where D , the dimension of the spacetime is-

$$D = 2n + \epsilon. \quad (41)$$

Here $\epsilon = 0, 1$ depending on whether D is even or odd. And this generates the towers of $n-1$ rank $(D-2j)$ Killing-Yano tensors:

$$Y^{(j)} = *F^{(j)} \quad (42)$$

and $n-1$ rank-2 Stäckel-Killing tensors:

$$K_{\mu\nu}^{(j)} = Y_{\mu\rho_1\dots\rho_{D-2j-1}}^{(j)} Y_{\nu}^{(j)\rho_1\dots\rho_{D-2j-1}}. \quad (43)$$

Now let's turn our attention to the spinning point particle in the presence of torsion. The rank-2 quantum phase space Dirac-type operator in the presence of torsion is given in [27]. Here, however, we are going to focus on writing the associated generalized Killing-Yano with the Stäckel-Killing tensors given by (38), (39), which we are going to notate with ω_1, ω_2 and the anticommutator of the Dirac-type operators associated with them. So the two generalized GKY tensors are:

$$\omega_1 = (\partial_\psi) \wedge \omega + (\partial_\psi) \wedge \tilde{\omega} - (\partial_\psi) \wedge \omega^\epsilon \quad (44)$$

and

$$\omega_2 = (\partial_\phi) \wedge \omega + (\partial_\phi) \wedge \tilde{\omega} - (\partial_\phi) \wedge \omega^\epsilon. \quad (45)$$

According to [27], rel. (4.19) it follows that the associated generalized Dirac-type operator with torsion is given by:

$$K_\omega = \omega^a{}_b \gamma^b \nabla_a + \frac{1}{18} (d\omega)_{abc} \gamma^{abc} - \frac{1}{24} T^a{}_{bc} \omega_{ad} \gamma^{bcd} - \frac{1}{4} T^{ab}{}_c \omega_{ab} \gamma^c \quad (46)$$

Consequently the anticommutator of two of the three Dirac-type operators associated with GKY tensors is (circular permutations between the three operators give similar relations):

$$\begin{aligned} \{K_1, K_2\} &= \gamma^{b_1} \gamma^{d_1} (\omega_1)^a{}_{b_1} (\omega_2)^c{}_{d_1} (2\nabla_c \nabla_a + R^e{}_{ace}) + \\ &+ \frac{3}{2} \gamma^{b_1} \gamma^{d_1} T^{cd}{}_{d_1} (\omega_1)^a{}_{b_1} (\omega_2)_{cd} \nabla_a + \frac{1}{9} \gamma^{d_1 d_2 d_3} \gamma^{b_1} (d\omega_2)_{d_1 d_2 d_3} (\omega_1)^a{}_{b_1} \nabla_a + \\ &+ \frac{1}{9} \gamma^{d_1 d_2 d_3} \gamma^{b_1} (d\omega_1)_{d_1 d_2 d_3} (\omega_2)^a{}_{b_1} \nabla_a - \frac{1}{12} \gamma^{b_1} \gamma^{d_1} T^{cd}{}_{d_1} (\omega_1)^a{}_{b_1} (d\omega_2)_{acd} - \\ &- \frac{1}{12} \gamma^{b_1} \gamma^{d_1} T^{cd}{}_{d_1} (d\omega_1)_{acd} (\omega_2)^a{}_{b_1} + \frac{5}{9} \gamma^{b_1 b_2 b_3} \gamma^{d_1 d_2 d_3} (d\omega_1)_{b_1 b_2 b_3} (d\omega_2)_{d_1 d_2 d_3} + \\ &\frac{1}{48} \gamma^{b_1 b_2 b_3} \gamma^{d_1 d_2 d_3} T^a{}_{b_1 b_2} T^c{}_{d_1 d_2} (\omega_1)_{ab_3} (\omega_2)_{cd_3} \quad (47) \end{aligned}$$

The three GKY tensors given by relations (44), (45) and the Hodge dual of (31) generate a hyper-Kähler deformed by torsion structure and the associated modified Dirac-type operators generate the superalgebra in rel. (47). A similar situation, but in a torsionless case is described in [19], where three Killing-Yano tensors (the square roots of the metric tensor) generate a hyper-Kähler structure and the associated Dirac-type operators generate a superalgebra for that type of spacetime, where the metric is a Stäckel-Killing tensor.

5 Killing spinors in D=5 minimal gauged supergravity

It is well-known that there is a close intertwining between the existence of Killing spinors and Killing-Yano tensors and other structures and that Killing spinors have been widely used to classify solutions of supergravity in various dimensions. The supersymmetric solutions of minimal gauged and ungauged supergravity were classified in [39, 40]. In particular for D=5 the Killing spinors obey the equation:

$$[D_\alpha + \frac{1}{4\sqrt{3}} (\gamma_\alpha^{\beta\gamma} - 4\delta_\alpha^\beta \gamma^\gamma) F_{\beta\gamma}] \epsilon^a - \chi \epsilon^{ab} (\frac{1}{4\sqrt{3}} \gamma_\alpha - \frac{1}{2} A_\alpha) \epsilon^b = 0, \quad (48)$$

where χ is a real constant and ϵ^{ab} the Levi-Civita tensor.

We are trying to go deeper here and explore the relationship between Killing spinors and Killing-Yano tensors and we are going to focus on a spacetime in 5 dimensions with deformed Sasaki structure with torsion in the context of minimal gauged supergravity. There has been a lot of work on Sasaki-Einstein manifolds, which are complete, compact and with positive Ricci curvature [53]. In 5 dimensions it is a well-known fact that Sasaki-Einstein manifolds admit Killing spinors [46]. From the pioneering work of Bochner and later Lichnerowicz who showed that the Dirac operators are invertable on compact manifolds of positive scalar curvature, to more recent work on generalized Killing spinors in 5 dimensions [52] there has been devoted in the course of time a lot of attention to solving the Dirac equation on curved spacetime. We place our work in this diverse and intense context in a specific 5 dimensional minimal gauged supergravity with deformed by torsion Sasaki structure we explicitly construct the Killing spinors.

We can now turn to solutions of eq.(56) for the case of a manifold with Sasaki deformed by torsion structure in 5-dimensional minimal gauged supergravity. In D=5 minimal gauged supergravity there exists always a symplectic Majorana spinor [41] and we are going to construct a solution to equation (56) on the model of the solution given in [42], where Killing spinors of the following form were constructed in AdS_5 :

$$\epsilon_i = (e^{\frac{i}{2}arM\gamma_r})_{ij}(\delta_{jk} + \frac{i}{2}ax^\alpha\gamma_\alpha(M_{jk} - i\delta_{jk}\gamma_r))\xi_k \quad (49)$$

With this starting point, we formulated our own ansatz that the Killing spinor that verifies equation (56) is of the form:

$$\epsilon_i = (e^{\frac{i}{2}\gamma^i x_i M})_j^k (\delta_i^j x^\alpha (\gamma_\alpha^{\beta\delta} - \delta_\alpha^\beta \gamma^\delta) F_{\beta\delta} + \frac{i\epsilon^{jl}}{2} \chi x^\alpha \gamma_\alpha (M_{il} - i\delta_{il} A_\alpha \gamma^\alpha)) \xi_k \quad (50)$$

where $M = \vec{x}\vec{\sigma}$ with $\vec{\sigma}$ the Pauli matrices and

$$\vec{x} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \quad (51)$$

and ϵ^{ij} is the Levi-Civita tensors and ξ_k is a symplectic Majorana spinor. Also please note that the vector x_α is given by (in the coordinates in which our Sasaki deformed by torsion minimal gauged supergravity metric is given):

$$x_0 = \psi_0 \quad (52)$$

$$x_1 = x\cos\theta \quad (53)$$

$$x_2 = x\sin\theta\cos\phi \quad (54)$$

$$x_3 = x \sin \theta \sin \phi \cos \psi_1 \quad (55)$$

$$x_4 = x \sin \theta \sin \phi \sin \psi_1 \quad (56)$$

Moreover ϵ_i needs to verify rel (5.1) in ref. [39], the integrability conditions of the Killing spinor and consequently there will be constraints for the Majorana spinors as well, if we plug in the expression for ϵ_i in rel. (5.1) of [39]. So the integrability condition reads:

$$\begin{aligned} & \left\{ \frac{1}{8} {}^5 R_{\rho\mu\nu_1\nu_2} \gamma^{\nu_1\nu_2} + \frac{1}{4\sqrt{3}} (\gamma_{[\mu}^{\nu_1\nu_2} + 4\gamma^{\nu_1} \delta_{[\mu}^{\nu_2]}) \nabla_{\rho]} F_{\nu_1\nu_2} - \right. \\ & - \frac{1}{48} (-2F^2 \gamma_{\mu\rho} - 8F_{\nu[\rho}^2 \gamma_{\mu]}^{\nu} + 12F_{\mu\nu_1} F_{\rho\nu_2} \gamma^{\nu_1\nu_2} + 8F_{\nu_1\nu_2} F_{\nu_3[\rho} \gamma_{\mu]}^{\nu_1\nu_2\nu_3}) - \\ & \left. - \frac{\chi^2}{48} \gamma_{\rho\mu} \} \epsilon^a - \frac{\chi}{24} (\gamma_{\rho\mu}^{\nu_1\nu_2} F_{\nu_1\nu_2} - 4F_{\nu[\rho} \gamma_{\mu]}^{\nu} - F_{\rho\mu}) \epsilon^{ab} \epsilon^b = 0 \quad (57) \end{aligned}$$

We work with the metric in [49], which is Sasaki with torsion and satisfies the equations of motion of 5-dimensional minimal gauged supergravity:

$$\begin{aligned} g = & (\xi - x)(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{dx^2}{Q(x)} + Q(x)(d\psi_1 + \cos \theta d\phi)^2 + \\ & + 4(d\psi_0 + (x + \frac{q}{x - \xi})d\psi_1 + (x - \xi + \frac{q}{x - \xi})\cos \theta d\phi)^2 \quad (58) \end{aligned}$$

where

$$Q(x) = \frac{4x^3 + (1 - 12\xi)x^2 + (8q - 2\xi + 12\xi^2)x + k}{\xi - x} \quad (59)$$

and q , ξ and k are free parameters. As it is well known the action in minimal gauged supergravity in 5 dimensions is:

$$\mathcal{L}_5 = *(\mathcal{R} - \Lambda) - \frac{1}{2} F \wedge *F + \frac{1}{3\sqrt{3}} F \wedge F \wedge A \quad (60)$$

where $F = dA$ and in our case the Maxwell potential is:

$$A = -\frac{2\sqrt{3}q}{x - \xi} (d\psi_1 + \cos \theta d\phi) \quad (61)$$

and the torsion $T = *F/\sqrt{3}$. The equations of motion are:

$$R_{ab} = -4g_{ab} + \frac{1}{2}(F_{ac}F_b{}^c - \frac{1}{6}g_{ab}F_{cd}F^{cd}) \quad (62)$$

$$d * F - \frac{1}{\sqrt{3}}F \wedge F = 0 \quad (63)$$

Consequently, we can derive these results for this metric:

$$F = \frac{2\sqrt{3}q}{(x-\xi)^2}dx \wedge d\psi_1 + \frac{2\sqrt{3}q}{x-\xi}\sin\theta d\theta \wedge d\phi + \frac{2\sqrt{3}q}{(x-\xi)^2}\cos\theta dx \wedge d\phi \quad (64)$$

The inverse metric has the following non-zero components:

$$g^{\psi_0\psi_0} = \frac{(\xi-x)}{D(\psi_0, \psi_1, x, \theta, \phi)}[(\xi-x)\sin^2\theta - 4Q(x)] \quad (65)$$

$$g^{\psi_1\psi_1} = \frac{4(\xi-x)^2\sin^2\theta}{Q(x)D(\psi_0, \psi_1, x, \theta, \phi)} \quad (66)$$

$$\begin{aligned} g^{xx} = & \frac{(\xi-x)}{D(\psi_0, \psi_1, x, \theta, \phi)} \{-16Q^2(x)\cos^2\theta + 4Q(x)[(\xi-x)\sin^2\theta + \\ & + 32\cos\theta(x + \frac{q}{x-\xi})(x-\xi + \frac{q}{x-\xi}) - 16(x-\xi + \frac{q}{x-\xi})^2] - \\ & - 320(x + \frac{q}{x-\xi})^2(x-\xi + \frac{q}{x-\xi})^2 - 48(x + \frac{q}{x-\xi})^2(\xi-x)\sin^2\theta\} \end{aligned} \quad (67)$$

$$\begin{aligned} g^{\theta\theta} = & \frac{1}{Q(x)D(\psi_0, \psi_1, x, \theta, \phi)} \{-8Q^2(x)\cos^2\theta + Q(x)[4(\xi-x)\sin^2\theta + \\ & + 128\cos\theta(x-\xi + \frac{q}{x-\xi})(x + \frac{q}{x-\xi}) - 64(x-\xi + \frac{q}{x-\xi})^2] - \\ & - 48(x + \frac{q}{x-\xi})^2(\xi-x)\sin^2\theta + 256(x + \frac{q}{x-\xi})^2(x-\xi + \frac{q}{x-\xi})^2\} \end{aligned} \quad (68)$$

$$g^{\phi\phi} = \frac{4(\xi-x)}{D(\psi_0, \psi_1, x, \theta, \phi)Q(x)}\{Q(x) - 48(x + \frac{q}{x-\xi})^2\} \quad (69)$$

$$\begin{aligned} g^{\psi_0\psi_1} = g^{\psi_1\psi_0} = & \frac{(\xi-x)}{Q(x)D(\psi_0, \psi_1, x, \theta, \phi)} \{-16Q(x)\cos\theta(x-\xi + \frac{q}{x-\xi}) + \\ & + (x + \frac{q}{x-\xi})(\xi-x)\sin^2\theta - 64(x + \frac{q}{x-\xi})(x-\xi + \frac{q}{x-\xi})^2\} \end{aligned} \quad (70)$$

$$g^{\psi_0 x} = g^{x\psi_0} = 0 \quad g^{\psi_0 \theta} = g^{\theta\psi_0} = 0 \quad (71)$$

$$g^{\psi_0 \phi} = g^{\phi\psi_0} = \frac{x - \xi}{Q(x)D(\psi_0, \psi_1, x, \theta, \phi)} \left\{ -16Q(x)\cos\theta(x - \xi + \frac{q}{x - \xi}) - \right. \\ \left. - 64(x - \xi + \frac{q}{x - \xi})^2(x + \frac{q}{x - \xi}) + 8(x + \frac{q}{x - \xi})(\xi - x)\sin^2\theta \right\} \quad (72)$$

$$g^{\psi_1 x} = g^{x\psi_1} = 0 \quad g^{\psi_1 \theta} = g^{\theta\psi_1} = 0 \quad (73)$$

$$g^{\psi_1 \phi} = g^{\phi\psi_1} = \frac{x - \xi}{Q(x)D(\psi_0, \psi_1, x, \theta, \phi)} \left\{ 8Q(x)\cos\theta - 32(x + \frac{q}{x - \xi})(x - \xi + \frac{q}{x - \xi}) \right\} \quad (74)$$

$$g^{x\theta} = g^{\theta x} = 0 \quad g^{x\phi} = g^{\phi x} = 0 \quad g^{\theta\phi} = g^{\phi\theta} = 0 \quad (75)$$

where the function D (the determinant of g) is:

$$D(\psi_0, \psi_1, x, \theta, \phi) = -16Q^2(x)\cos^2\theta + Q(x)[4(\xi - x)\sin^2\theta + \\ + 224(x + \frac{q}{x - \xi})(x - \xi + \frac{q}{x - \xi})\cos\theta - 64(x - \xi + \frac{q}{x - \xi})^2] + \\ + 16(x + \frac{q}{x - \xi})^2[-3(\xi - x)\sin^2\theta + 32(x - \xi + \frac{q}{x - \xi})] \quad (76)$$

And the torsion tensor is:

$$*F = g^{xx}g^{\psi_1\psi_1}\frac{2\sqrt{3}q}{(x - \xi)^2}d\psi_0 \wedge d\theta \wedge d\phi + g^{\theta\theta}g^{\phi\phi}\frac{2\sqrt{3}q}{x - \xi}\sin\theta d\psi_0 \wedge d\psi_1 \wedge dx + \\ + g^{xx}g^{\phi\phi}\frac{2\sqrt{3}q}{(x - \xi)^2}\cos\theta d\psi_0 \wedge d\psi_1 \wedge d\theta. \quad (77)$$

The inverse vielbeins are:

$$\hat{e}_1 = -g^{\theta\theta}\sqrt{\xi - x}d\theta \quad (78)$$

$$\hat{e}_2 = g^{\phi\phi}\sqrt{\xi - x}\sin\theta d\phi + g^{\psi_1\phi}\sqrt{\xi - x}\sin\theta d\psi_1 + g^{\psi_0\phi}\sqrt{\xi - x}\sin\theta d\psi_0 \quad (79)$$

$$\hat{e}_3 = g^{xx} \frac{1}{\sqrt{Q(x)}} dx \quad (80)$$

$$\begin{aligned} \hat{e}_4 = & (g^{\psi_1\psi_1} \sqrt{Q(x)} + g^{\psi_1\phi} \sqrt{Q(x)} \cos\theta) d\psi_1 + \\ & + (g^{\psi_0\psi_1} \sqrt{Q(x)} + g^{\psi_0\phi} \sqrt{Q(x)} \cos\theta) d\psi_0 + \\ & + (g^{\phi\psi_1} \sqrt{Q(x)} + g^{\phi\phi} \sqrt{Q(x)} \cos\theta) d\phi \end{aligned} \quad (81)$$

$$\begin{aligned} \hat{e}_5 = & 2(g^{\psi_1\psi_0} + g^{\psi_1\psi_1}(x + \frac{q}{x-\xi}) + g^{\psi_1\phi}(x - \xi + \frac{q}{x-\xi})) d\psi_1 + \\ & + 2(g^{\psi_0\psi_0} + g^{\psi_0\psi_1}(x + \frac{q}{x-\xi}) + g^{\psi_0\phi}(x - \xi + \frac{q}{x-\xi})) d\psi_0 + \\ & + 2(g^{\phi\psi_0} + g^{\phi\psi_1}(x + \frac{q}{x-\xi}) + g^{\phi\phi}(x - \xi + \frac{q}{x-\xi})) d\phi \end{aligned} \quad (82)$$

And now in preparation of calculating the Ricci tensor and then the Riemann tensor, we get for F^{ab} in vielbein indices:

$$\begin{aligned} F = & \frac{2\sqrt{3}q}{x-\xi} (\frac{1}{x-\xi} + \frac{3}{2Q(x)}) \hat{e}^1 \wedge \hat{e}^4 - \frac{2\sqrt{3}q}{(x-\xi)Q(x)} \hat{e}^3 \wedge \hat{e}^4 + \\ & + (-\frac{\sqrt{3}q \cos\theta \sin\theta}{\sqrt{Q(x)}(\xi-x)^{3/2}}) \hat{e}^3 \wedge \hat{e}^2 + \frac{2\sqrt{3}q \sin^2\theta}{(x-\xi)^2} \hat{e}^1 \wedge \hat{e}^2 \end{aligned} \quad (83)$$

And now we know using our metric that the only not-null components of the Ricci tensor are:

$$R_{aa} = -4g_{aa} + \frac{1}{2}(g_{aa}^2 g_{cc} F^{ac} F^{ac} - \frac{1}{6} g_{aa} g_{cc} g_{dd} F^{cd} F^{cd}) \quad (84)$$

Detailing this with 4 indices (note that $g_{aa}^{\mu\nu} = e_a^\mu e_a^\nu$):

$$\begin{aligned} R_{55}^{\mu\nu} = & -4g_{55}^{\mu\nu} - \frac{1}{3} g_{55}^{\mu\nu} \{ g_{11} g_{44} (F^{14})^2 + g_{11} g_{22} (F^{12})^2 + g_{33} g_{44} (F^{34})^2 + \\ & + 2g_{33} g_{22} (F^{32})^2 \} \end{aligned} \quad (85)$$

$$\begin{aligned}
R_{44}{}^{\mu\nu} &= -4g_{44}^{\mu\nu} + \frac{1}{2}\{(g_{44}^2)^{\mu\nu}(g_{11}(F^{41})^2 + g_{33}(F^{43})^2) - \\
&- \frac{1}{3}g_{44}^{\mu\nu}\{g_{11}g_{44}(F^{14})^2 + g_{11}g_{22}(F^{12})^2 + g_{33}g_{44}(F^{34})^2 + 2g_{33}g_{22}(F^{32})^2\}\} \quad (86)
\end{aligned}$$

$$\begin{aligned}
R_{33}{}^{\mu\nu} &= -4g_{33}^{\mu\nu} + \frac{1}{2}\{(g_{33}^2)^{\mu\nu}(g_{44}(F^{34})^2 + g_{22}(F^{32})^2) - \\
&- \frac{1}{3}g_{33}^{\mu\nu}\{g_{11}g_{44}(F^{14})^2 + g_{11}g_{22}(F^{12})^2 + g_{33}g_{44}(F^{34})^2 + 2g_{33}g_{22}(F^{32})^2\}\} \quad (87)
\end{aligned}$$

$$\begin{aligned}
R_{22}{}^{\mu\nu} &= -4g_{22}^{\mu\nu} + \frac{1}{2}\{(g_{22}^2)^{\mu\nu}(g_{33}(F^{23})^2 + g_{11}(F^{21})^2) - \\
&- \frac{1}{3}g_{22}^{\mu\nu}\{g_{11}g_{44}(F^{14})^2 + g_{11}g_{22}(F^{12})^2 + g_{33}g_{44}(F^{34})^2 + 2g_{33}g_{22}(F^{32})^2\}\} \quad (88)
\end{aligned}$$

$$\begin{aligned}
R_{11}{}^{\mu\nu} &= -4g_{11}^{\mu\nu} + \frac{1}{2}\{(g_{11}^2)^{\mu\nu}(g_{22}(F^{12})^2 + g_{44}(F^{14})^2) - \\
&- \frac{1}{3}g_{11}^{\mu\nu}\{g_{11}g_{44}(F^{14})^2 + g_{11}g_{22}(F^{12})^2 + g_{33}g_{44}(F^{34})^2 + 2g_{33}g_{22}(F^{32})^2\}\} \quad (89)
\end{aligned}$$

And now using the formula:

$$R_{\rho\sigma}{}^{\mu\nu} = e_\rho^a e_\sigma^a R_{aa}{}^{\mu\nu} \quad (90)$$

we finally get to the Riemann tensor we need in the integrability conditions stated above:

$$R_{\rho\sigma\alpha\beta} = g_{\alpha\mu}g_{\beta\nu}R_{\rho\sigma}{}^{\mu\nu} \quad (91)$$

This whole context was useful to find the Riemann and the electromagnetic tensors, which appear in the integrability conditions on the Killing spinor. In the end these integrability conditions translate in constraints on the Majorana spinors and the fact that $\chi^2 = 1$.

We now turn to determining the Christoffel symbols and the spin connection coefficients. The only not null Christoffel coefficients are:

$$\Gamma_{xx}^x = -\frac{1}{2Q(x)^2}g^{xx}\{-8x^3 - (1-24\xi)x^2 + 2\xi(1-12\xi)x + (8q-2\xi+12\xi^2)\xi + k\}. \quad (92)$$

$$\begin{aligned} \Gamma_{\psi_1 x}^{\psi_0} &= 4g^{\psi_0 \psi_0} \left(1 - \frac{q}{(x-\xi)^2}\right) + \frac{1}{2}g^{\psi_0 \psi_1}[-8x^3 - (1-24\xi)x^2 + 2\xi(1-12\xi)x + \\ &\quad + 8(x + \frac{q}{x-\xi})(1 - \frac{q}{(x-\xi)^2}) + (8q-2\xi+12\xi^2)\xi + k] \end{aligned} \quad (93)$$

$$\begin{aligned} \Gamma_{x\psi_1}^{\psi_0} &= 4g^{\psi_0 \psi_0} \left(1 - \frac{q}{(x-\xi)^2}\right) + \frac{1}{2}g^{\psi_0 \psi_1}[-8x^3 - (1-24\xi)x^2 + 2\xi(1-12\xi)x + \\ &\quad + 8(x + \frac{q}{x-\xi})(1 - \frac{q}{(x-\xi)^2}) + (8q-2\xi+12\xi^2)\xi + k] + \frac{1}{2}g^{\psi_0 \phi}[2(-8x^3 - (1-24\xi)x^2 + \\ &\quad + 2\xi(1-12\xi)x) + 8(x + \frac{q}{x-\xi})(2x - \xi + \frac{2q}{x-\xi}) + 2[(8q-2\xi+12\xi^2)\xi + k]] \end{aligned} \quad (94)$$

$$\Gamma_{x\psi_0}^{\psi_1} = 4(g^{\psi_1 \psi_1} + g^{\psi_1 \phi})\left(1 - \frac{q}{(x-\xi)^2}\right) \quad (95)$$

$$\Gamma_{\psi_0 x}^{\psi_1} = 4(g^{\psi_1 \psi_1} + g^{\psi_1 \psi_0} + g^{\psi_1 \phi})\left(1 - \frac{q}{(x-\xi)^2}\right) \quad (96)$$

$$\Gamma_{\psi_1 \psi_0}^x = \Gamma_{\psi_0 \psi_1}^x = -4g^{xx}\left(1 - \frac{q}{(x-\xi)^2}\right) \quad (97)$$

$$\begin{aligned} \Gamma_{x\phi}^{\psi_0} &= 4g^{\psi_0 \psi_0} \left(1 - \frac{q}{(x-\xi)^2}\right) + \frac{1}{2}g^{\psi_0 \psi_1}[-2\cos\theta(-8x^3 - (1-24\xi)x^2 + \\ &\quad + 2\xi(1-12\xi)x) + 8(1 - \frac{q}{(x-\xi)^2})(2x - \xi + \frac{2q}{x-\xi}) + 2\cos\theta[(8q-2\xi+12\xi^2)\xi + k]] \end{aligned} \quad (98)$$

$$\begin{aligned}
\Gamma_{\phi x}^{\psi_0} &= 4g^{\psi_0\psi_0}\left(1 - \frac{q}{(x-\xi)^2}\right) + \frac{1}{2}g^{\psi_0\psi_1}[-2\cos\theta(-8x^3 - (1-24\xi)x^2 + \\
&+ 2\xi(1-12\xi)x) + 8(1 - \frac{q}{(x-\xi)^2})(2x - \xi + \frac{2q}{x-\xi}) + 2\cos\theta[(8q - 2\xi + 12\xi^2)\xi + k]] - \\
&- \frac{1}{2}g^{\psi_0\phi}\sin^2\theta \quad (99)
\end{aligned}$$

$$\Gamma_{\psi_0\phi}^x = \Gamma_{\phi\psi_0}^x = -4g^{xx}\left(1 - \frac{q}{(x-\xi)^2}\right) \quad (100)$$

$$\begin{aligned}
\Gamma_{\psi_0x}^{\phi} &= 4g^{\phi\psi_0}\left(1 - \frac{q}{(x-\xi)^2}\right) + \frac{1}{2}g^{\phi\psi_1}[-2\cos\theta(-8x^3 - (1-24\xi)x^2 + \\
&+ 2\xi(1-12\xi)x) + 8(1 - \frac{q}{(x-\xi)^2})(2x - \xi + \frac{2q}{x-\xi}) + 2\cos\theta[(8q - 2\xi + 12\xi^2)\xi + k]] - \\
&- \frac{1}{2}g^{\phi\phi}\sin^2\theta \quad (101)
\end{aligned}$$

$$\Gamma_{x\psi_0}^{\phi} = 4(g^{\phi\psi_1} + g^{\phi\phi})\left(1 - \frac{q}{(x-\xi)^2}\right) \quad (102)$$

$$\begin{aligned}
\Gamma_{x\phi}^{\psi_1} &= 4g^{\psi_1\psi_1}\left(1 - \frac{q}{(x-\xi)^2}\right) + \frac{1}{2}g^{\psi_0\psi_1}[-2\cos\theta(-8x^3 - (1-24\xi)x^2 + \\
&+ 2\xi(1-12\xi)x) + 8(1 - \frac{q}{(x-\xi)^2})(2x - \xi + \frac{2q}{x-\xi}) + 2\cos\theta[(8q - 2\xi + 12\xi^2)\xi + k]] - \\
&- \frac{1}{2}g^{\psi_1\phi}\sin^2\theta \quad (103)
\end{aligned}$$

$$\begin{aligned}
\Gamma_{\psi_1 x}^\phi &= \Gamma_{x \psi_1}^\phi = 4g^{\phi\psi_0}\left(1 - \frac{q}{(x-\xi)^2}\right) + \frac{1}{2}g^{\phi\phi}[-2\cos\theta(-8x^3 - (1-24\xi)x^2 + \\
&+ 2\xi(1-12\xi)x) + 8(1 - \frac{q}{(x-\xi)^2})(2x - \xi + \frac{2q}{x-\xi}) + 2\cos\theta[(8q - 2\xi + 12\xi^2)\xi + k]] + \\
&+ \frac{1}{2}g^{\psi_1\phi}[-8x^3 - (1-24\xi)x^2 + 2\xi(1-12\xi)x + (8q - 2\xi + 12\xi^2)\xi + k + 8(x + \frac{q}{x-\xi}) \\
&\quad (1 - \frac{q}{(x-\xi)^2})] \quad (104)
\end{aligned}$$

The Dirac operator is written as:

$$D_\mu\psi = \partial_\mu\psi + \omega_{\mu ab}\Sigma^{ab}\psi \quad (105)$$

where the spin connections can be calculated as:

$$\omega_\mu{}^a{}_b = -e^a{}_\lambda e_b{}^\kappa \Gamma_{\mu\kappa}^\lambda - e^a{}_\sigma \partial_\mu e_b{}^\sigma \quad (106)$$

Note that in the end we need to calculate the following coefficients and actually only 30 are not null, out of which we need to calculate only 15, given the fact that the coefficients are antisymmetric in ab:

$$\omega_{\mu ab} = g_{aa}(-e^a{}_\lambda e_b{}^\kappa \Gamma_{\mu\kappa}^\lambda - e^a{}_\sigma \partial_\mu e_b{}^\sigma) \quad (107)$$

We are going to write the extended formulae (which are non-trivial) for only three of them and we are going to leave the rest in a constrained form. Here they are:

$$\begin{aligned}
\omega_{\psi_0 43} &= -g^{xx} \Gamma_{\psi_0 x}^{\psi_1} = \frac{-4(\xi - x)^2}{Q(x) D^2(\psi_0, \psi_1, x, \theta, \phi)} \{-16Q^2(x) \cos^2 \theta + \\
&+ 4Q(x) [(\xi - x) \sin^2 \theta + 32 \cos \theta (x + \frac{q}{x - \xi})(x - \xi + \frac{q}{x - \xi}) - 16(x - \xi + \frac{q}{x - \xi})^2] - \\
&- 320(x + \frac{q}{x - \xi})^2 (x - \xi + \frac{q}{x - \xi})^2 - 48(x + \frac{q}{x - \xi})^2 (\xi - x) \sin^2 \theta\} \\
&\{-8Q(x) \cos \theta (2x - 2\xi + \frac{2q}{x - \xi} + 1) + (x + \frac{q}{x - \xi})(\xi - x) \sin^2 \theta - 4(\xi - x) \sin^2 \theta \\
&- 64(x + \frac{q}{x - \xi})(x - \xi + \frac{q}{x - \xi})^2 - 32(x + \frac{q}{x - \xi})(x - \xi + \frac{q}{x - \xi})\} (1 - \frac{q}{(x - \xi)^2}) \\
&\quad (108)
\end{aligned}$$

$$\begin{aligned}
\omega_{\psi_0 53} &= -\frac{-8(\xi - x)^2}{Q(x)^{3/2} D^2(\psi_0, \psi_1, x, \theta, \phi)} \{-16Q^2(x) \cos^2 \theta + \\
&+ 4Q(x) [(\xi - x) \sin^2 \theta + 32 \cos \theta (x + \frac{q}{x - \xi})(x - \xi + \frac{q}{x - \xi}) - 16(x - \xi + \frac{q}{x - \xi})^2] - \\
&- 320(x + \frac{q}{x - \xi})^2 (x - \xi + \frac{q}{x - \xi})^2 - 48(x + \frac{q}{x - \xi})^2 (\xi - x) \sin^2 \theta\} \\
&\{-8Q(x) \cos \theta (2x - 2\xi + \frac{2q}{x - \xi} + 1) + (x + \frac{q}{x - \xi})(\xi - x) \sin^2 \theta - \\
&- 4(\xi - x) \sin^2 \theta - 64(x + \frac{q}{x - \xi})(x - \xi + \frac{q}{x - \xi})^2 - \\
&- 32(x + \frac{q}{x - \xi})(x - \xi + \frac{q}{x - \xi})\} (1 - \frac{q}{(x - \xi)^2}) (x + \frac{q}{x - \xi}) \quad (109)
\end{aligned}$$

$$\begin{aligned}
\omega_{\psi_0 32} = & \frac{16(\xi - x)^{\frac{7}{2}} \sin \theta}{D^3(\psi_0, \psi_1, x, \theta, \phi) Q^2(x)} \{ -16Q^2(x) \cos^2 \theta + 4Q(x)[(\xi - x) \sin^2 \theta + \\
& 32 \cos \theta (x + \frac{q}{x - \xi})(x - \xi + \frac{q}{x - \xi}) - 16(x - \xi + \frac{q}{x - \xi})^2] - \\
& - 320(x + \frac{q}{x - \xi})^2(x - \xi + \frac{q}{x - \xi})^2 - 48(x + \frac{q}{x - \xi})^2(\xi - x) \sin^2 \theta \} \\
& \{ -8Q(x) \cos \theta (2x - 2\xi + \frac{2q}{x - \xi} + 1) + (x + \frac{q}{x - \xi})(\xi - x) \sin^2 \theta - \\
& - 4(\xi - x) \sin^2 \theta - 64(x + \frac{q}{x - \xi})(x - \xi + \frac{q}{x - \xi})^2 - \\
& - 32(x + \frac{q}{x - \xi})(x - \xi + \frac{q}{x - \xi}) \} (1 - \frac{q}{x - \xi})^2 \quad (110)
\end{aligned}$$

And now the rest of the spin connections in constrained form:

$$\omega_{\psi_1 53} = -\frac{2g^{xx}}{\sqrt{Q(x)}} \Gamma_{\psi_1 x}^{\psi_0} \quad (111)$$

$$\omega_{\psi_1 34} = -\frac{g^{xx}}{Q(x)} (g^{\psi_0 \psi_1} + g^{\psi_0 \phi} \cos \theta) \Gamma_{\psi_1 \psi_0}^x \quad (112)$$

$$\omega_{\psi_1 32} = -\frac{\sqrt{\xi - x} \sin \theta}{Q(x)^{\frac{3}{2}}} g^{xx} g^{\psi_0 \phi} \Gamma_{\psi_1 \psi_0}^x \quad (113)$$

$$\omega_{\phi 53} = -\frac{2g^{xx}}{\sqrt{Q(x)}} \Gamma_{\phi x}^{\psi_0} \quad (114)$$

$$\omega_{\phi 34} = -\frac{g^{xx}}{Q(x)} Q(x) (g^{\psi_0 \psi_1} + g^{\psi_0 \phi} \cos \theta) \Gamma_{\phi \psi_0}^x \quad (115)$$

$$\omega_{\phi 32} = -\frac{\sqrt{\xi - x} \sin \theta}{Q(x)^{\frac{3}{2}}} g^{xx} g^{\psi_0 \phi} \Gamma_{\phi \psi_0}^x \quad (116)$$

$$\begin{aligned}
\omega_{x54} = & 2\sqrt{Q(x)}\left(x + \frac{q}{x-\xi}\right)(g^{\psi_0\psi_1} + g^{\psi_0\phi}\cos\theta)\Gamma_{x\psi_0}^{\psi_1} + \\
& + \{16(g^{\psi_0\psi_0} + g^{\psi_0\psi_1}\left(x + \frac{q}{x-\xi}\right) + g^{\psi_0\phi}\left(x - \xi + \frac{q}{x-\xi}\right))^2 + \\
& + 4(g^{\phi\psi_0} + g^{\phi\psi_1}\left(x + \frac{q}{x-\xi}\right) + g^{\phi\phi}\left(x - \xi + \frac{q}{x-\xi}\right))^2(\xi - x)\sin^2\theta + \\
& + 4(g^{\psi_1\psi_0} + g^{\psi_1\psi_1}\left(x + \frac{q}{x-\xi}\right) + g^{\psi_1\phi}\left(x - \xi + \frac{q}{x-\xi}\right))^2[Q(x) + 4\left(x + \frac{q}{x-\xi}\right)^2] + \\
& + 32(g^{\psi_1\psi_0} + g^{\psi_1\psi_1}\left(x + \frac{q}{x-\xi}\right) + g^{\psi_1\phi}\left(x - \xi + \frac{q}{x-\xi}\right)) \\
& (g^{\psi_0\psi_0} + g^{\psi_0\psi_1}\left(x + \frac{q}{x-\xi}\right) + g^{\psi_0\phi}\left(x - \xi + \frac{q}{x-\xi}\right))\left(x + \frac{q}{x-\xi}\right) + \\
& + 32(g^{\psi_0\psi_0} + g^{\psi_0\psi_1}\left(x + \frac{q}{x-\xi}\right) + g^{\psi_0\phi}\left(x - \xi + \frac{q}{x-\xi}\right)) \\
& (g^{\phi\psi_0} + g^{\phi\psi_1}\left(x + \frac{q}{x-\xi}\right) + g^{\phi\phi}\left(x - \xi + \frac{q}{x-\xi}\right))\left(x - \xi + \frac{q}{x-\xi}\right)\} \\
& \{(g^{\psi_1\psi_1}\sqrt{Q(x)} + g^{\psi_1\phi}\sqrt{Q(x)}\cos\theta) + (g^{\phi\psi_1}\sqrt{Q(x)} + \\
& + g^{\phi\phi}\sqrt{Q(x)}\cos\theta)\cos\theta\}\left(1 - \frac{q}{(x-\xi)^2}\right) \quad (117)
\end{aligned}$$

We write the following two in constrained form noting that the term in g_{55} writes the same way in both the two related terms:

$$\begin{aligned}
\omega_{x52} = & \sqrt{\xi - x}\sin\theta\left\{-2\left(x + \frac{q}{x-\xi}\right)g^{\psi_0\phi}\Gamma_{x\psi_0}^{\psi_1} + \right. \\
& \left. + e_5^\mu e_5^\nu g_{\mu\nu}(g^{\phi\phi} + g^{\psi_1\phi})\left(1 - \frac{q}{(x-\xi)^2}\right)\right\} \quad (118)
\end{aligned}$$

$$\begin{aligned}\omega_{x42} = & \sqrt{\xi - x} \sin\theta \{ \sqrt{Q(x)} g^{\psi_0\phi} \Gamma_{x\psi_0}^{\psi_1} + \\ & + \frac{\partial_x Q(x)}{2\sqrt{Q(x)}} e_4^\mu e_4^\nu g_{\mu\nu} (g^{\psi_1\phi} + g^{\phi\phi} \cos\theta) \} \quad (119)\end{aligned}$$

The last three spin connection coefficients are:

$$\omega_{\theta 24} = \sqrt{(\xi - x)Q(x)} \cos\theta e_2^\mu e_2^\nu g_{\mu\nu} (g^{\phi\psi_1} + g^{\phi\phi} \cos\theta) \quad (120)$$

$$\omega_{\theta 25} = 2\sqrt{\xi - x} \cos\theta [g^{\phi\psi_0} + g^{\phi\psi_1} (x + \frac{q}{x - \xi}) + g^{\phi\phi} (x - \xi + \frac{q}{x - \xi})] e_2^\mu e_2^\nu g_{\mu\nu} \quad (121)$$

$$\omega_{\theta 45} = -\sqrt{Q(x)} \sin\theta [g^{\phi\psi_0} + g^{\phi\psi_1} (x + \frac{q}{x - \xi}) + g^{\phi\phi} (x - \xi + \frac{q}{x - \xi})] e_4^\mu e_4^\nu g_{\mu\nu} \quad (122)$$

After highly non-trivial work and cancellations, plugging in formula (50) in eq. (56) gives the much sought-after null result.

To be noted the fact that the current eq. (56) differs from the equation of Killing spinors with torsion given in [48], in other words here we work with regular Killing spinors. As a matter of fact with the definition quoted in [48], in the same paper it is proved that there are no Killing spinors with torsion for 5-dimensional Sasaki-Einstein manifolds. It is a subtle distinction between these two mathematical objects, with different definitions on different manifolds, and this concludes our proof here.

To be noted also that the finding of the Killing spinors is equivalent to finding the nilpotent orbits of the associated Lie groups of the respective supergravities[44,45] In our case the Lie group is $O(4,1) \times O(1,4)$ of $N=1$ $D=5$ minimal gauged supergravity. One can extend the study of the relationship between the existence of a (generalized) Killing-Yano tensor and that of a Killing spinor to other supergravities in various dimensions, but we leave this for future work.

6 Conclusions

We overviewed briefly some of the symmetries of the Kerr-Newman, $D=5$ minimal gauged supergravity endowed with a Killing-Maxwell system and higher dimensional spacetimes with Killing-Yano torsion, to find that if a Killing-Maxwell electromagnetic field is present, it becomes the PCKY or the PGCKY in the cases with torsion and that it generates towers of Killing-Yano and Stäckel-Killing tensors of the spacetime. For the studied 5-dimensional spacetime the Hodge dual of the Killing-Maxwell electromagnetic field plays the role of torsion

and is at the same time a generalized Killing-Yano tensor of the spacetime, being derived naturally from the PGCKY. Some results regarding Killing spinors for the 5-dimensional minimal gauged supergravity are derived also.

The generalized Killing-Yano tensors together with covariant derivatives form Dirac-type operators which are not anomalous and characterize additional supersymmetries of the spinning point particle in curved spacetimes with Killing-Yano torsion. It is very interesting that these supersymmetries are generated by the Killing-Maxwell electromagnetic field when this is present and this sparks further investigation of the correlation between the electromagnetic gauge symmetry and supersymmetry in curved spacetimes. Also another interesting track would be to determine the dual (torsionless) spacetime of the D=5 minimal gauged supergravity spacetime. Moreover, it would be interesting to see whether other Killing-Yano tensors exist in the studied supergravity spacetime (of dimensionality 5) except for the Hodge dual of the Killing-Maxwell electromagnetic field.

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